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one above the plane $\zeta = 1$ and to the right of the plane $x = 0$, the other below the plane $\zeta = 1$ and to the left of $x = 0$. Each sheet of the surface approaches the plane $\zeta = 1$ asymptotically.[1] Every section of the surface by a plane parallel to the z-plane is a circle. The nature of the essential singularity at $z = 0$ is well illustrated.

(4) $\zeta = |\sin z|$ and (5) $\zeta = |\sin^2 z|$. The Cartesian equations of these surfaces are easily found to be $\zeta = \sqrt{\sin^2 x + \sinh^2 y}$ and $\zeta = \sin^2 x + \sinh^2 y$. The surfaces show the periodic character of the functions and the fact that $|\sin z|$ may take on any positive value in a period strip.

These examples are intended only to illustrate and not to exhaust the possibilities of the subject. The student should try to visualize the surface $\zeta = |\sin 1/z|$ in the neighborhood of its essential singularity. A few hours spent in constructing these surfaces will contribute towards overcoming the feeling of "a stranger in a strange land" which most students experience in entering upon the study of the theory of functions of a complex variable.

MODULAR GEOMETRY.

By ALBERT A. BENNETT, Baltimore, Md.

(Read before the Maryland-District of Columbia-Virginia Section of the Mathematical Association of America, May 15, 1920.)

The study of relations in a single variable, in the case of finite number fields, is not, of course, a new idea, although it is awakening perhaps more interest now than in the last century. A study of several variables and in particular a characteristically geometrical treatment for finite fields may fairly be said to have been introduced within the past few years, and the only extended discussions of the whole subject to be found in any general treatise are those in Veblen and Young's *Projective Geometry,* and in a little book by G. Arnoux, *Essai de Géométrie Analytique Modulaire,* Paris, 1911. References are given in L. E. Dickson, *On Invariants and the Theory of Numbers* (Madison Colloquium Lectures), 1914, page 98.

§ 1. Plane Geometry, Modulo 3.

We shall for the sake of concreteness, consider the individual case of the field, modulo 3. This may be thought of as generated by the set of residues obtained by dividing integers by the number 3. Thus any integer, n, falls in one of three mutually exclusive classes according as n, $n - 1$, or $n - 2$ is an integral multiple of 3. We say that n is respectively congruent to 0, 1 or 2, expressed in symbols by $n \equiv 0$ (*mod* 3), $n \equiv 1$ (*mod* 3), $n \equiv 2$ (*mod* 3), respectively. There are then only three numbers, 0, 1, 2, in the field. For these we have the following addition and multiplication tables.

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[1] Except in the $y\zeta$-plane. The plane $\zeta = 1$ intersects the surface in a line parallel to the y-axis.—Editor.

Addition:

	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

Multiplication:

	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

Subtraction and division (except by 0) are always uniquely possible. Incidentally $-1 = \frac{1}{2} = 2$, in this field. The equation $x(x-1)(x-2) = 0$, has 0, 1, 2, as roots, and therefore $x^3 - x = 0$ is identically satisfied by every number of the field. Every linear equation with coefficients in the field has a unique root in the field, but quadratic equations exist with coefficients in the field whose roots are imaginary with respect to the field, that is, are not themselves in the field but may be introduced by a consistent extension of the system. Thus $x^2 + 1 = 0$, is not satisfied by 0, 1, or 2, and hence has no real roots in the field. Every polynomial with coefficients in the field may be successively reduced if of degree higher than 2 by means of the identity $x^3 = x$, so as to be represented as of degree not greater than 2. The following are therefore the totality of essentially distinct polynomials in the field:

$$0,\quad 1,\quad 2,\quad x^2,\quad x^2+1,\quad x^2+2,\quad 2x^2,\quad 2x^2+1,\quad 2x^2+2,$$

$$x,\quad x+1,\quad x+2,\quad x^2+x,\quad x^2+x+1,\quad x^2+x+2,\quad 2x^2+x,\quad 2x^2+x+1,\quad 2x^2+x+2,$$

$$2x,\quad 2x+1,\quad 2x+2,\quad x^2+2x,\quad x^2+2x+1,\quad x^2+2x+2,\quad 2x^2+2x,\quad 2x^2+2x+1,\quad 2x^2+2x+2.$$

One may proceed in this manner to study functions of one variable. But our present interest is geometry, taken for simplicity as of two dimensions, so that we shall not inquire further into properties of polynomials in one variable.

By points in the non-homogeneous modular plane (*mod* 3) will be meant any set of objects each of which is regarded as determining uniquely and being uniquely determined by an ordered pair of numbers of the modular field. The number of distinct points is equal to the number of distinct ordered pairs, (m, n), where each of these is in the field. The plane consists therefore of nine points. By a straight line in the plane is meant the set of points satisfying a single linear equation in two unknowns, the whole being in the field. Each line is found to contain three points. The set of distinct lines corresponds to the following, and are twelve in number.

$$x = 0,\qquad y = 0,\qquad x + y = 0,\qquad x + 2y = 0,$$

$$x = 1,\qquad y = 1,\qquad x + y = 1,\qquad x + 2y = 1,$$

$$x = 2,\qquad y = 2,\qquad x + y = 2,\qquad x + 2y = 2.$$

The equation $2x + y = 1$, for example, is identical in the field with $x + 2y = 2$, either equation being obtained from the other by multiplying by 2. Through any point there will be four distinct lines of the plane. Lines are either parallel or intersect in a point of the plane as in the usual system. For example,

$2x + y = 1$, and $x + y = 2$, intersect in a point whose coördinates may be obtained by the usual method of determinants, viz.,

$$\left(\frac{\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix}}{\begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix}}, \frac{\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix}}{\begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix}} \right) = (2, 0).$$

Quadratic loci may be studied in the usual manner. In particular, there are circles. These are of three classes according as the square of the radius is 0, 1 or 2. A circle with imaginary radius, *e.g.*, $\sqrt{2} = \sqrt{-1}$, has in this geometry real points. For example, $x^2 + y^2 = 2$, contains the points (1, 2), (2, 1), (1, 1), (2, 2), which are as numerous as the points on the circle, $x^2 + y^2 = 1$, which has a real radius, namely the points, (1, 0), (2, 0), (0, 1), (0, 2). The circle of zero radius contains as usual but one point. Thus the three circles with any given common center, together contain all the points of the plane, or in other words a circle may be drawn with a given center and passing through a given point. It is interesting to notice that the circle of radius one is also of radius two, since in this system $2 = -1$.

The trigonometry in this plane is rather primitive since the only angles occurring are multiples of 45 degrees. The trigonometric functions yield the usual table reduced modulo 3.

	sin	cos	tan	cot	sec	csc
0°	0	1	0	–	1	–
45°	$2i$	$2i$	1	1	i	i
90°	1	0	–	0	–	1
135°	$2i$	i	2	2	$2i$	i
180°	0	2	0	–	2	–
225°	i	i	1	1	$2i$	$2i$
270°	2	0	–	0	–	2
315°	i	$2i$	2	2	i	$2i$

where however i is not a real number but may be introduced as an imaginary quantity satisfying the relation $i^2 = 2 = -1$.

Ellipses, parabolas, hyperbolas, all exist with their usual properties although many of these become trivial. Higher plane curves do not exist in the real plane.

As in the ordinary geometry, real points fall into three classes with respect to a non-degenerate circle: those points from which there are two real distinct tangents,[1] those with but one real tangent, those with no real tangents to the circle. Ordinarily the first is the class of exterior points, the second, the class of points on the circle, and the third the class of interior points of the circle.

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[1] By tangent we mean the line whose equation is formed as in the ordinary analytical geometry. For example, the tangent to the circle $x^2 + y^2 = 1$ at a point x', y' will be the line whose equation is $xx' + yy' = 1$.

Were this adopted as a definition we would say that for every non-degenerate circle in this modular plane there is one interior point, the center, four real points on the circle and four exterior points. Unfortunately for intuition, the four points exterior by this definition to a circle, C_1, are on a circle, C_2, with the same center, while the four points on C_1 are by this definition themselves exterior to C_2. The line joining an exterior point with the center of the circle does not meet the circle in real points.

§ 2. Imaginaries in Modular Geometries.

Whereas for ordinary plane geometry the introduction of all imaginaries may be effected in one step, this is not true in the same sense for modular geometries. This is a consequence of the failure in modular geometries of the following theorem, valid for the numbers of ordinary analysis: While an algebraic equation with real coefficients may fail to have a real root, the system of complex numbers, $a + ib$, where a and b are real and i is a square root of minus one, is such that every algebraic equation with complex coefficients (of degree greater than zero), has a complex root (real numbers being included in the term, complex).

This is of course the so-called fundamental theorem of algebra. It must be distinguished from the sub-case frequently employed which states that every algebraic equation with real coefficients (of degree greater than zero) has a root which is expressible in the form $a + ib$.

For a modular geometry a theorem analogous to this sub-theorem is still true, but the general theorem first stated must be much extended. Thus if in a modular geometry an algebraic equation with real coefficients be given, it is always possible to introduce a single "imaginary" quantity, i, such that the given equation has a root of the form $a + ib$, where a and b are real. Unfortunately the i depends upon the degree of the original equation, assumed irreducible, and a new equation with coefficients involving i will require a further extension of the system. The system requires an infinite number of steps of this sort before it is complete.

We have mentioned only the Euclidean analogue, but the classical non-Euclidean geometries and the more general projective geometry have also corresponding modular geometries.

§ 3. Significance of Modular Geometry.

The study of modular geometry is justified from several points of view. In the first place, an investigator drilled in the concepts of general algebraic geometry is better capable of distinguishing in a given problem the essential from the non-essential features, and will seek recourse in complicated relations only when these are inherent in the problem. By avoiding extraneous machinery, proofs are secured which are at the same time economical and of wide applicability.[1]

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[1] The points of a given Euclidean space being assumed, and the notion of dimensionality being established, one may distinguish among the remaining geometric notions two categories having very little beyond the concepts of point and dimension in common. These may be roughly identified with the terms "continuity" and "algebraic rationality." In the one case there is a body

In the second place finite groups, in particular multiply transitive groups can regularly be interpreted and frequently with profit, in the geometrical language of a suitably chosen modular space. The abstract group becomes in many cases geometrically intuitive only by reference to an interpretation of this sort. In the third place the subject still presents novelty and has interest on its own account, the methods of investigation are extremely simple, and most of the results for a given modular space are obtained with ease and completeness.

QUESTIONS AND DISCUSSIONS.

EDITED BY W. A. HURWITZ, Cornell University, Ithaca, N. Y.

REPLIES.

15 [*1914*, 278; *1916*, 353; *1920*, 114]. In the *Proceedings of the Royal Society of Edinburgh*, Vol. 7, p. 144, in some mathematical notes by Professor P. G. Tait, it is stated:

"If $x^3 + y^3 = z^3$, then $(x^3 + z^3)^3y^3 + (x^3 - y^3)^3z^3 = (z^3 + y^3)^3x^3$.

"This furnishes an easy proof of the impossibility of finding two integers the sum of whose cubes is a cube."

How does this "easy proof" follow?

REMARKS BY THE EDITOR.

A number of communications relative to this question have been received. As regards content they fall into three classes. In the first (and largest) class may be placed mere demonstrations of the algebraic premiss of the question,—that if $x^3 + y^3 = z^3$, then

$$(x^3 + z^3)^3y^3 + (x^3 - y^3)^3z^3 = (z^3 + y^3)^3x^3.$$

This is indeed easy, and we are not to suppose that the question implied any difficulty in obtaining an identity within the reach of any high school student who has learned the fundamentals of algebraic notation. It seems necessary to emphasize, therefore, that the truth of the algebraic identity is granted, and that what is desired is an "easy" proof, based on this identity, that the equation $x^3 + y^3 = z^3$ has no solutions in positive integers.

In a second group fall the replies indicating that Tait may have been mistaken. One suggestion is that he may have noted that any of the relations $x = y$, $x = -z$, $y = -z$ will reduce the second equation to an identity, but not the first. This is not impossible; it amounts to supposing that Tait considered the two equations as equivalent,—that he regarded the second as being not only a necessary, but also a sufficient condition for the first. If so, he would not be the only mathematician of note who has confused necessary and sufficient conditions; but such an error would be so obvious in the present instance as to seem rather unlikely. Another correspondent suggests that Tait took for granted that an equation of the form $x^n + y^n = z^n$ cannot hold for an infinite number of sets of values of x, y, z when $n > 1$. Of course, however, this

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of definitions and theorems concerned only with continuity and forming a science which may be called the Analysis Situs of the given space; in the other case there is a corresponding science treating of algebraic relations among figures describable in geometrico-algebraic language.

By employing the term "algebra" in a wide sense to apply to commutative number fields (that is, number fields with commutative multiplication), one may study a "general algebraic geometry," which will include as sub-cases all those distinct instances that arise by further characterizing the commutative number field under consideration. The entire science may be regarded as the study, in geometrical language to be sure, of algebraic relations in a given number of variables defined over a general commutative number field. Theorems in this general algebraic geometry are clearly independent of Analysis Situs since, as has been shown, number fields may be selected in which continuity has no application.